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OPTIMAL CONTROL OF SOME BILINEAR SYSTEMS WITH AFTEREFFECT*

V.B. KOLMANOVSKII and N.I. KOROLEVA

The optimal control of bilinear systems with aftereffect is considered. A class of systems is identified for which the optimal control is constructed by solving linear differential equations. Recursive formulas are derived for this solution. As an example, we analyse a bilinear model with aftereffect of the process of bacterial growth in a microbiological controlled environment.

A system is usually called bilinear if the evolution equations are linear in the phase coordinates for fixed controls and in the controls for fixed coordinates /1, 2/. Systems of this kind are used for modelling a variety of control processes, including processes in biological systems /3-5/, etc.

1. Statement of the problem. Optimality conditions. For simplicity, we will initially consider controlled systems with one constant delay and a scalar control.

$$x^{\cdot}(t) = A_1(t)\,x(t-h) + (A(t)\,x(t) + B(t))u \qquad (1.1)$$
$$0 \leqslant t \leqslant T,\ x \in R_n,\ u \in R_1,\ h = \mathrm{const} \geqslant 0$$

Here $A(t)$ and $B(t)$ are matrices with given piecewise-continuous elements, $A_1(t)$ is a piecewise-continuously differentiable matrix. The solution of Eq.(1.1) is determined by the initial condition

$$x(\theta) = \varphi(\theta),\ -h \leqslant \theta \leqslant 0,\ \varphi(\theta) \in R_n \qquad (1.2)$$

where $\varphi(\theta)(-h \leqslant \theta \leqslant 0)$ is a given piecewise-continuous bounded function. Let D be the space of such functions. The control u in (1.1) should be designed as $u = u(t, \varphi)$ which is piecewise-continuous in t and continuous in $\varphi \in D$, minimizing the performance functional

$$x'(T)\,N_1 x(T) + \int_0^T [x'(t)\,N_2(t)\,x(t) + u'(t)\,N(t)\,u(t) + f(t, x_t)]\,dt;\quad N_i \geqslant 0,\quad N(t) > 0 \qquad (1.3)$$

Here x_t is the section of the trajectory $x(t+\theta)(-h \leqslant \theta \leqslant 0)$. In (1.3) the elements of the matrices $N_2(t)$ and $N(t)$ are piecewise-continuous and the form of the functional $f(t, x_t)$ is given below.

The choice of f in the functional (1.3) is based on the principle of generalized /6/ and the dynamic programming method /7/, as modified for systems with aftereffect. Let us describe this method.

Denote by $V(t, \varphi)(\varphi \in D)$ the Bellman functional of problem (1.1)-(1.3). The Bellman equation has the form

$$\inf_{u \in R_1} [V^{\cdot}(t, \varphi) + \varphi'(0) N_2(t) \varphi(0) + N(t) u^2 + f(t, \varphi)] = 0$$
$$V(T, \varphi) = \varphi'(0) N_1 \varphi(0) \qquad (1.4)$$

Here $V^{\cdot}(t, \varphi)$ is the total derivative of the functional V along the trajectory of system (1.1) for the value of the control parameter u occurring in (1.4). We stress that the infimum in (1.4) is over the scalar paraemter u.

The solution of the problem (1.4) is sought in the form

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*Prikl.Matem.Mekhan.,53,2,238-243,1989

$$V(t, \varphi) = \varphi'(0) P(t) \varphi(0) + \varphi'(0) \int Q(t, \tau) \varphi(\tau) d\tau + \int \varphi'(\tau) Q'(t, \tau) d\tau \varphi(0) + \int_{-h}^{0} \int_{-h}^{0} \varphi'(\tau) R(t, \tau, \rho) \varphi(\rho) d\tau d\rho \quad (1.5)$$

where P, Q and R are to be determined. Here and throughout Sect.1, integration over τ is between the limits $-h$ to 0. From (1.5) it follows that we should have $R(t, \tau, \rho) = R'(t, \rho, \tau)$.

Compute $V^{\cdot}$, substitute the result into (1.4), and replace $x^{\cdot}(t)$ from Eq.(1.1). This gives a control that minimizes the left-hand side of (1.4),

$$u_0(t, x_t) = -N^{-1}(t) [A(t) x(t) + B(t)]' \left[P(t) x(t) + \int Q(t, \tau) x(t + \tau) d\tau\right] \quad (1.6)$$

Now choose f so as to eliminate the non-linear terms in P, Q, R from Eq.(1.4). To this end, we should set

$$f = N u_0^2 \quad (1.7)$$

Substituting (1.5)-(1.7) into (1.4) and equating to zero the quadratic forms in $x(t)$ and $x(t+\theta)$, we obtain a system of linear differential equations for the matrices $P(t)$, $Q(t, \tau)$, $R(t, \tau, \rho)$:

$$\begin{gathered} P^{\cdot}(t) + Q(t, 0) + Q'(t, 0) + N_2(t) = 0 \\ (\partial/\partial t - \partial/\partial \tau) Q(t, \tau) + R(t, \tau, 0) = 0 \\ (\partial/\partial t - \partial/\partial \tau - \partial/\partial \rho) R(t, \tau, \rho) = 0 \\ 0 \leqslant t \leqslant T, \ -h \leqslant \tau, \rho \leqslant 0 \end{gathered} \quad (1.8)$$

Similarly equating to zero the quadratic forms in $x(t-h)$ and applying (1.4), we obtain the boundary conditions

$$\begin{gathered} P(T) = N_1, \ Q(T, \bar{\tau}) \equiv 0, \quad R(T, \bar{\tau}, \bar{\rho}) \equiv 0, \ -h < \bar{\tau}, \bar{\rho} \leqslant 0 \\ Q'(t, -h) = A_1'(t) P(t), \ R(t, \tau, \rho) = R'(t, \rho, \tau) \\ R(t, -h, \tau) + R'(t, \tau, -h) = 2A_1'(t) Q(t, \tau) \\ 0 \leqslant t \leqslant T, \ -h \leqslant \tau, \rho \leqslant 0 \end{gathered} \quad (1.9)$$

Note /8/ that in the class of piecewise-continuously differentiable bounded functions there exists a unique solution P, Q, R of the boundary value problem (1.8), (1.9) if the matrix $N_2(t)$ is bounded and piecewise-continuous and the matrix $A_1(t)$ is differentiable with respect to t and has piecewise-continuous bounded derivatives.

In order to prove the optimality of the control (1.6), we have to establish the existence of a solution $x^\circ(t)$ of problem (1.1), (1.2) for $u = u_0$ which is bounded in the interval $[0, T]$. Since the conditions of the local existence theorem /9/ are satisfied, it suffices to check the boundedness of the function $x^\circ(t)$.

By (1.4), and (1.7), we have $V(t, x_t^\circ) \leqslant V(0, \varphi)$. This and (1.5) give

$$|x(t)|^2 \leqslant V(0, \varphi) + c\left(|x(t)| \int |x(t+\tau)| d\tau + \left(\int |x(t+\tau)| d\tau\right)^2\right), \quad c = \text{const} > 0 \quad (1.10)$$

From (1.10), using the Cauchy inequality, we conclude that for any $\varepsilon > 0$,

$$|x(t)|^2 \leqslant V(0, \varphi) + c\left(\varepsilon h |x(t)|^2 + (h + \varepsilon^{-1}) \int |x(t+\tau)|^2 d\tau\right)$$

Taking $\varepsilon > 0$ so that $c\varepsilon h < 1$ and applying the Gronwall-Bellman lemma, we prove the existence of a bounded solution of problem (1.1), (1.2) in $[0, T]$.

Thus, the design of the optimal control (1.6) and the determination of the corresponding value of the performance criterion (1.5) have been reduced to the solution of problem (1.8) and (1.9).

*2. **Construction of the solution of problem (1.8) and (1.9).*** Let us describe the method of solving problem (1.8) and (1.9). Note that the value of the functional

$$J(t) = x'(T) N_1 x(T) + \int_t^T x'(s) N_2(s) x(s) ds \quad (2.1)$$

on the trajectories of the system

$$x^{\cdot}(s) = A_1(s) x(s-h), \ s \geqslant t; \ x(t+\theta) = \varphi(\theta), \ -h \leqslant \theta \leqslant 0 \quad (2.2)$$

is defined by (1.5).

First let us find $J(t)$ for $T-h \leqslant t \leqslant T$. In the case, $J(t)$ can be determined in two ways. On the one hand, for $t \geqslant T-h$, system (2.2) is without aftereffect and therefore $J(t)$ can be found analytically as a quadratic form of the initial conditions. On the other hand, $J(t)$ is defined by (1.5). Equating the values of $J(t)$ obtained in these different ways, we obtain expressions for P, Q, R for $T-h \leqslant t \leqslant T$. We then determine P, Q, R for $T-2h \leqslant t \leqslant T$. To this end, we represent $J(t)$ in the form

$$J(t) = \int_t^{T-h} x'(s) N_2(s) x(s)\, ds + J_1$$

$$J_1 = x'(T) N_1 x(T) + \int_{T-h}^{T} x'(s) N_2(s) x(s)\, ds$$

and substitute (1.5) for J_1, using the matrices P, Q, R obtained in the previous step for $T-h \leqslant t \leqslant T$. We thus obtain that for $T-2h \leqslant t \leqslant T$ the functional $J(t)$ is also a quadratic form of the trajectory. Therefore, applying the same comparison technqiue, we find P, Q, R for $T-2h \leqslant t \leqslant T$. Proceeding along the same lines, we can determine the matrices P, Q, R for all $t \in [0, T]$.

We will now give the corresponding expressions.

Represent the solution of the boundary value problem (1.8), (1.9) as the sum of two solutions: the first solution for $N_2 = 0$ and an arbitrary matrix N_1, and the second solution for $N_1 = 0$ and an arbitrary matrix N_2. Direct substitution will show that the first solution ($N_2 = 0$) has the form

$$P(t) = B_2'(t) N_1 B_2(t), \quad Q(t, \tau) = -B_2'(t) N_1 B_2^{\cdot}(t+\tau)$$
$$R(t, \tau, \rho) = B_2^{\cdot\prime}(t+\tau) N_1 B_2^{\cdot}(t+\rho), \; 0 \leqslant t \leqslant T, \; -h \leqslant \tau, \rho \leqslant 0 \tag{2.3}$$

where the matrix $B_2(t)$ satisfies the relationships

$$B_2^{\cdot}(t) = -B_2(t+h) A_1(t+h), \quad 0 \leqslant t \leqslant T$$
$$B_2(s) = 0, \quad s > T; \quad B_2(T) = I \tag{2.4}$$

and at the point of discontinuity $T-h$ of the derivative $B_2^{\cdot}(t)$ it is defined by left continuity.

Let us give the recursive formulas for the second solution ($N_1 = 0$). Let $t_i = T - ih$, where $i = 0, 1, \ldots$. For $t_1 \leqslant t \leqslant T$, we have

$$P(t) = \int_t^T N_2(s)\, ds, \quad Q(t, \tau) = \int_{t+\tau+h}^{T} N_2(s)\, ds \cdot A_2(t+\tau+h)$$

$$R(t, \tau, \rho) = A_2'(t+\tau+h) \int_{t+h+\max(\tau,\rho)}^{T} N_2(s)\, ds \cdot A_2(t+\rho+h)$$

and the matrix A_2 for $t_{i+1} \leqslant t \leqslant t_i$ and $-h \leqslant \tau \leqslant 0$ is given by

$$A_2(t+\tau+h) = A_1(t+\tau+h), \quad -h \leqslant \tau \leqslant -t + t_{i+1}$$
$$A_2(t+\tau+h) = 0, \quad -t + t_{i+1} < \tau \leqslant 0$$

Now for $t_{i+1} \leqslant t \leqslant t_i$ let

$$Q_1(t_i, \tau+t-t_i) = \begin{cases} Q(t_i, \tau+t-t_i), & -t+t_{i+1} \leqslant \tau \leqslant 0 \\ 0, & -h \leqslant \tau < -t+t_{i+1} \end{cases}$$

Similarly define $R_1(t_i, \tau+t-t_i, \rho)$ as a function of τ for any fixed ρ.

For $t_{i+1} \leqslant t \leqslant t_i$, $-t+t_{i+1} \leqslant \tau, \rho \leqslant 0$, we have the equality

$$R_2(t_i, \tau+t-t_i, \rho+t-t_i) = R(t_i, \tau+t-t_i, \rho+t-t_i)$$

If at least one of the arguments τ or ρ does not belong to the interval $[-t+t_{i+1}, 0]$, then $R_2(t_i, \tau+t-t_i, \rho+t-t_i) = 0$.

Now for $t_{i+1} \leqslant t \leqslant t_i$, we have

$$P(t) = \int_t^{t_i} N_2(s)\,ds + P(t_i) + \int_{t-t_i}^{0} (Q(t_i, s) + Q'(t_i, s))\,ds +$$
$$\int_{t-t_i}^{0}\int_{t-t_i}^{0} R(t_i, s, \alpha)\,ds\,d\alpha \qquad (2.5)$$

The matrix Q satisfies the equality

$$Q(t, \tau) = \Big[\int_{t+\tau+h}^{t_i} N_2(s)\,ds + P(t_i) + \int_{t-t_i}^{0} Q(t_i, s)\,ds +$$
$$\int_{t-t_{i+1}+\tau}^{0} Q(t_i, s)\,ds + \int_{t-t_i}^{0} d\alpha \int_{t-t_{i+1}+\tau}^{0} R(t_i, s, \alpha)\,ds\Big] A_2(t+\tau+h) +$$
$$Q_1(t_i, t+\tau-t_i) + \int_{t-t_i}^{0} R_1(t_i, t+\tau-t_i, s)\,ds, \quad t_{i+1} \leqslant t \leqslant t_i, \; -h \leqslant \tau \leqslant 0 \qquad (2.6)$$

Finally, for R we have

$$R(t, \tau, \rho) = A_2'(t+\tau+h)\Big[\int_{t+h+\max(\tau,\rho)}^{t_i} N_2(s)\,ds + P(t_i) +$$
$$\int_{t-t_{i+1}+\rho}^{0} Q(t_i, s)\,ds + \int_{t-t_{i+1}+\tau}^{0} Q(t_i, s)\,ds +$$
$$\int_{t-t_{i+1}+\tau}^{0} d\alpha \int_{t-t_{i+1}+\rho}^{0} R(t_i, s, \alpha)\,ds\Big] A_2(t+\rho+h) +$$
$$A_2(t+\tau+h)\Big[Q_1(t_i, t+\rho-t_i) + \int_{t+\tau-t_{i+1}}^{0} R_1(t_i, s, t+\rho-t_i)\,ds\Big] +$$
$$\Big[Q_1(t_i, t+\tau-t_i) + \int_{t+\rho-t_{i+1}}^{0} R_1(t_i, t+\tau-t_i, s)\,ds\Big] A_2(t+\rho+h) +$$
$$R_2(t_i, t+\tau-t_i, t+\rho-t_i) \qquad (2.7)$$

The recursive formulas (2.5)-(2.7) define the second solution of the boundary value problem (1.8) and (1.9) required.

3. Some Generalizations.

Let us generalize our results to a controlled system of the form

$$x^{\cdot}(t) = A_0(t)x(t) + A_1(t)x(t-h) + \int_{-h}^{0} G(t, s)x(t+s)\,ds + B(t)u +$$
$$\sum_{j=1}^{m} u_j\Big[A_{3j}(t)x(t) + A_{4j}(t)x(t-h_j) +$$
$$\int_{-\tau_j}^{0} G_{1j}(t, s)x(t+s)\,ds\Big], \quad t > 0; \quad x(t) \in R_n, \quad u \in R_m \qquad (3.1)$$

The elements of the matrices $A_0, G, B, A_{3j}, A_{4j}, G_{1j}$ are given piecewise-continuous functions, the elements of the matrix A_1 are piecewise-continuously differentiable functions, and h, h_j and τ_j are given non-negative constants.

The functional being minimized has the form (1.3), and the initial condition is (1.2). Let b_j, $(j = 1, \ldots, m)$ be the columns of the matrix B. Consider the $n \times m$ matrix $F(t, x_t)$ with the j-th column

$$F_j(t, x_t) = b_j(t) + A_{3j}(t)x(t) + A_{4j}(t)x(t-h_j) + \int_{-\tau_j}^{0} G_{1j}(t, s)x(t+s)\,ds \qquad (3.2)$$

Using this notation, we rewrite Eq.(3.1) in the form

$$x^{\cdot}(t) = A_0(t)\,x(t) + A_1(t)\,x(t-h) + \int_{-h}^{0} G(t, s)\,x(t+s)\,ds + F(t, x_t)\,u \tag{3.3}$$

Modifying the analysis of Sect.1 to allow for the particular form of Eq.(3.1), we conclude that the optimal control is given by

$$u_0(t, x_t) = -N^{-1}(t)\,F'(t, x_t)\left[P(t)\,x(t) + \int_{-h}^{0} Q(t, \tau)\,x(t+\tau)\,d\tau\right] \tag{3.4}$$

The optimal value of the functional (1.3) for

$$f(t, x_t) = u_0'(t, x_t)Nu_0(t, x_t)$$

is given by (1.5). The matrices P, Q, R in (3.4), (1.5) satisfy the linear equations

$$\begin{gathered} P^{\cdot}(t) + P(t)A(t) + A'(t)P(t) + Q'(t, 0) + Q(t, 0) + \\ N_2(t) = 0,\ 0 \leqslant t \leqslant T \\ P(t)G(t,\tau) + A'(t)Q(t,\tau) + R(t,0,\tau) + (\partial/\partial t - \partial/\partial\tau)Q(t,\tau) = 0 \\ G'(t, \tau)Q(t, \rho) + Q'(t, \tau)G(t, \rho) + (\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R(t, \tau, \rho) = 0 \end{gathered} \tag{3.5}$$

The boundary conditions for system (3.5) remain as before (1.9). Note that although the optimal control (3.4) and the trajectory depend on the delays h_j and τ_j, the optimal value of the functional (1.5) for $f = u_0'Nu_0$ is independent of the delays. In particular, it follows that for $h = 0$ the optimal value of the functional (3.2) is independent of the initial function $\varphi(s)$ for $s < 0$ and is determined entirely by the value of $\varphi(0)$, while both the trajectory and the control essentially depend on the initial function φ. As in Sect.2, we can write the solution of the boundary value problem (3.5), (1.9) for $G \equiv 0$ in a form similar to (2.3), (2.5)-(2.7).

4. *Example.*

Consider the control of bacterial growth and the formation of a microbiological product in a closed vessel. Continuous reproduction of micro-organisms is used in bacteriological research, in ferment production, and in biological effluent processing /10, 11/.

A certain mass of active bacteria is placed in a vessel equipped with an entrance for nutrients and an exit for extracting the product. The bacteria consume the nutrients, produce the output product during a finite length of time, reproduce, and then lose their viability.

This process can be described by a bilinear model of the form

$$\begin{aligned} m^{\cdot}(t) &= \gamma(t)\,m(t) - u(t)\,m(t) - m(t-\tau) \\ s^{\cdot}(t) &= -\gamma(t)\,k_1^{-1}m(t) - u(t)\,s(t) + s_r u(t) \end{aligned} \tag{4.1}$$

The first equation describes the biomass balance in the closed vessel, the second characterizes the synthesis of the product. Here, $m(t)$ is the volume of the microbiological mass, $s(t)$ is the output product volume, $u(t)$ is the nutrient volume, $\gamma(t)$ is the rate of growth of the bacteria, $m(t-\tau)$ allows for the finite active lifetime of the bacteria τ, and k_1 and s_r are constants.

Initially, at t_0,

$$s(t_0) = 0,\ m(t_0) = m_0,\ m(t_0 + \theta) = 0,\ -\tau \leqslant \theta < 0 \tag{4.2}$$

The problem is to achieve a fixed volume of the output product s_0 in a finite time while minimizing the consumption of nutrients. The performance criterion for this problem is taken in the form

$$J = \beta_1(s(T) - s_0)^2 + \beta_2 m(T)^2 + \int_{t_0}^{T} ((s(t) - s_0)^2 + \alpha u^2(t) + f(t, s, m_t))\,dt \tag{4.3}$$

Fig.1 plots the phase trajectories for $m(t)$ and $s(t)$ under the optimal control constructed by the proposed method.

The numerical solution of the problem was obtained for the following parameter values: $t_0 = 0$, $T = 3$, $\tau = 1$, $s_r = 3.5$, $s_0 = 1.5$, $k_1 = 52$, $\gamma(t) = 0.05$, $\beta_1 = \beta_2 = 1$, $\alpha = 1$, $m_0 = 4$.

The graphs show that s tends to s_0 and m tends to zero.

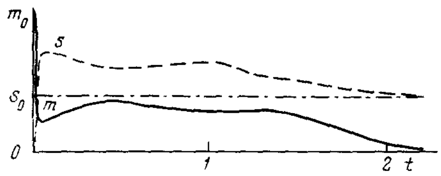

Fig.1

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